Exercises Towards Collapsing PH if Graph Isomorphism is NP-complete
CSCI 6114 Fall 2023

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The $\mathsf{BP}\cdot$ complexity class operator is defined as follows; $L \in \mathsf{BP}\cdot\mathcal{C}$ if there is a language $L_V \in \mathcal{C}$ ("$_V$" for "verifier") and a polynomial p such that for all strings x,

$$x \in L \Rightarrow Pr_{r\in\{0,1\}^{p(n)}}[(x,r) \in L_V] \geq 3/4$$
$$x \notin L \Rightarrow Pr_{r\in\{0,1\}^{p(n)}}[(x,r) \in L_V] \leq 1/4$$

Recall from last week, you showed:

1. $\mathsf{co}\cdot\mathsf{BP}\cdot\mathcal{C} = \mathsf{BP}\cdot\mathsf{co}\cdot\mathcal{C}$
2. If $\mathcal{C}$ is closed under majority reductions (which both P and NP are), then $\mathsf{BP}\cdot\mathsf{BP}\cdot\mathcal{C} = \mathsf{BP}\cdot\mathcal{C}$.
3. If $\mathcal{C}$ is closed under majority reductions, then $\exists\cdot\mathsf{BP}\cdot\mathcal{C} \subseteq \mathsf{BP}\cdot\exists\cdot\mathcal{C}$. (On last week's homework, to review in class.)

In-Class Exercises Tuesday

1. (a) Show that if $\mathsf{NP} \subseteq \mathsf{BPP}$, then $\mathsf{PH} \subseteq \mathsf{BPP}$. *Hint:* Use induction and Properties 2 and 3 above.
   (b) Show that if $\mathsf{NP} \subseteq \mathsf{coAM}$, then $\mathsf{PH} \subseteq \mathsf{AM}$. *Hint:* Similar to previous part, playing around with complexity class operators.

In-Class Exercises Thursday

In class, I'll teach the proof that if $\mathcal{C}$ is closed under majority reductions, then $\mathsf{BP}\cdot\mathcal{C} \subseteq \forall\cdot\exists\cdot\mathcal{C}\cap\exists\cdot\forall\cdot\mathcal{C}$. This will be needed for the following exercises.

2. Together with the result from lecture and the exercises from Tuesday's class, show that $\mathsf{NP} \subseteq \mathsf{BPP} \Rightarrow \mathsf{PH} = \mathsf{BPP}$ (note "=" not just "$\subseteq$" in the conclusion) and $\mathsf{NP} \subseteq \mathsf{coAM} \Rightarrow \mathsf{PH} = \mathsf{AM}$ (note both that it is "=" and that it is AM not coAM in the conclusion).
3. (a) Show that $\mathsf{AM} \subseteq \Pi_2\mathsf{P}$. (Note: why don't we also get $\mathsf{AM} \subseteq \Sigma_2\mathsf{P}$?)
   (b) Show that $\mathsf{MA} \subseteq \Sigma_2\mathsf{P} \cap \Pi_2\mathsf{P}$. (Note how much stronger this is than merely $\mathsf{BPP} \subseteq \Sigma_2\mathsf{P} \cap \Pi_2\mathsf{P}$!)
4. Prove that if Graph Isomorphism is NP-complete, then PH collapses to the second level.

Resources

- Our approach to this material is closest to that in Chapter 2 of Köbler, Schöning, and Torán.
- Arora & Barak Section 8.4.3 covers the result about NP-completeness of GI implying collapse of PH. The rest of their Chapter 8 covers interactive proofs more generally. (Note: they have an exercise to show that $\mathsf{AM} \subseteq \Sigma_3\mathsf{P}$, but in fact we will see it is contained in $\Sigma_2\mathsf{P}$.)
- The result about GI is also covered in Homer & Selman Section 10.5. The rest of Chapter 10 covers probabilistic classes such as BPP (and friends); their Section 12.3 covers Arthur–Merlin games, while the rest of their Chapter 12 covers more general interactive proofs.
- Zachos, S. Probabilistic quantifiers and games, *J. Comput. Syst. Sci.* 36(3):433–451, 1988. doi:10.1016/0022-0000(88)90037-2
- **Warning!** There is another class called "$\exists\mathsf{BPP}$", but it is not the same as $\exists \cdot \mathsf{BPP} = \mathsf{MA}$. See the Complexity Zoo entry for the difference (it has to do with how the randomized machines behave on all witnesses vs how they behave only on the one accepted witness).